Inverse Theory Cheat Sheet

Linear Inverse Problems

Definition of the problem:

$$\mathbf{Am} = \mathbf{d}$$

Where $\mathbf{A}$ is the forward problem, $\mathbf{m}$ is the model, and $\mathbf{d}$ is the data vector. $\mathbf{A}$ and $\mathbf{d}$ are known, and $\mathbf{m}$ is not.
Rescale using covariance matrix:

$$\begin{aligned}\hat{\mathbf{A}} &= \mathbf{C^{1/2}A} \\ \hat{\mathbf{d}} &= \mathbf{C^{1/2}d}\end{aligned}$$

We want:

$$\hat{\varepsilon} \equiv \hat{\mathbf{d}} - \hat{\mathbf{A}}\mathbf{m} \rightarrow \mathbf{0}$$

and

$$\nu\mathbf{Lm} \rightarrow \mathbf{0}$$

Find $\mathbf{m}$ that minimizes

$$\hat{\varepsilon}^T\hat{\varepsilon} + \nu^2(\mathbf{Lm})^T(\mathbf{Lm})$$

This requirement is satisfied by finding $\mathbf{m}$ using the following equation:

$$\begin{bmatrix}\hat{\mathbf{A}} \\ \nu\mathbf{L}\end{bmatrix}\mathbf{m} = \begin{bmatrix}\hat{\mathbf{d}} \\ \mathbf{0}\end{bmatrix}$$

The solution for $\mathbf{m}$ is:

$$\mathbf{m} = (\hat{\mathbf{A}}^T\hat{\mathbf{A}} + \nu^2\mathbf{L}^T\mathbf{L})^{-1}(\hat{\mathbf{A}}\hat{\mathbf{d}} + \nu\mathbf{L}^T\mathbf{0})$$

(Weakly) Non-linear Inverse Problems

In this case we make an initial guess $\mathbf{m_o}$:

$$\mathbf{m_o} \Rightarrow \mathbf{m} = \mathbf{m_o} + \delta\mathbf{m}$$

The forward problem is defined by

$$\begin{aligned}d_i &= F_i(\mathbf{m}) \\ d_i &= F_i(\mathbf{m} + \delta\mathbf{m})\end{aligned}$$

F_i is expanded using the first term of a Taylor series expansion:

$$d_i = F_i(\mathbf{m_o}) + \frac{\partial F_i}{\partial m_j}\delta m_j$$

$$\delta d \equiv \mathbf{d} - \mathbf{d}^o = \mathbf{A}^o\delta\mathbf{m}$$

where $\mathbf{d}$ is the real data, and $\mathbf{d}^o$ is the data predicted by $\mathbf{m}_o$.
Re-scaling:

$$\begin{aligned}\hat{\mathbf{A}}^o &= \mathbf{C}^{-1/2}\mathbf{A}^o \\ \delta\hat{\mathbf{d}} &= \mathbf{C}^{-1/2}\delta\mathbf{d}\end{aligned}$$

Scaled version:

$$\hat{\mathbf{A}}^o\delta\mathbf{m} = \delta\hat{\mathbf{d}}$$

We want:

$$\hat{\varepsilon} \equiv \delta\hat{\mathbf{d}} - \mathbf{A}^o\delta\mathbf{m} \rightarrow \mathbf{0}$$

and

$$\nu\mathbf{Lm} \rightarrow \mathbf{0}$$

What is $\mathbf{m}$?

$$\nu\mathbf{L}(\mathbf{m}_o + \delta\mathbf{m}) \rightarrow \mathbf{0}$$
$$\nu\mathbf{L}\delta\mathbf{m} = -\nu\mathbf{L}\mathbf{m}_o$$

$$\begin{bmatrix}\hat{\mathbf{A}}^\mathbf{o} \\ \nu\mathbf{L}\end{bmatrix}\delta\mathbf{m} = \begin{bmatrix}\delta\hat{\mathbf{d}} \\ -\nu\mathbf{L}\mathbf{m}_o\end{bmatrix}$$

The solution for $\delta\mathbf{m}$ is :

$$\delta\mathbf{m} = (\hat{\mathbf{A}}^{oT}\hat{\mathbf{A}}^o + \nu^2\mathbf{L}^T\mathbf{L})^{-1}(\hat{\mathbf{A}}^{oT}\delta\hat{\mathbf{d}} - \nu^2\mathbf{L}^T\mathbf{L}\mathbf{m}_o)$$

$$\mathbf{m}^{new} = \mathbf{m}_o + \delta\mathbf{m} \tag{1}$$

Now start again, substituting $\mathbf{m}^{new}$ for $\mathbf{m}_o$. Iterate until $\delta\mathbf{m}$ is suffiently small (depends on the problem).